

INTEGRABILITY OF THE HIDE-SKELDON-ACHESON DYNAMO

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ABSTRACT. In this work we consider the Hide-Skeldon-Acheson dynamo model

$$\dot{x} = x(y - 1) - \beta z, \quad \dot{y} = \alpha(1 - x^2) - \kappa y, \quad \dot{z} = x - \lambda z,$$

where α, β, κ and λ are parameters. We contribute to the understanding of its global dynamics, or more precisely, to the topological structure of its orbits by studying the integrability problem. Provided $\alpha \neq 0$ we identify the values of the parameters of this model, for which it admits a first integral. Also, as corollary of our main results we get that for $\alpha, \beta, \kappa \neq 0$ the dynamo model does not admit a polynomial, rational or Darboux first integral.

1. INTRODUCTION

In 1996 Hide, Skeldon and Acheson [6] proposed a model for self-exciting dynamo action in which a Faraday disk and coil are arranged in series with either a capacitor or a motor. The governing equations for these dynamo models are

$$(1) \quad \dot{x} = x(y - 1) - \beta z, \quad \dot{y} = \alpha(1 - x^2) - \kappa y, \quad \dot{z} = x - \lambda z,$$

where $\alpha, \beta, \kappa, \lambda$ are real parameters. In what follows those models will be called *HSA dynamo*. In (1) $x = x(t)$ is the current flowing through the dynamo, $y = y(t)$ is the angular rotation rate of the disk and $z = z(t)$ is the angular rotation rate of the motor (or the charge in the capacitor). The system contains four parameters α, β, κ and λ . The first one α is proportional to the steady applied mechanical couple driving the disk into rotation, and β^{-1} measures the moment of inertia of the armature of the motor, and κ and λ are the coefficients of friction in the disk and the motor, respectively. For the derivations of those equations see [6]. Since the HSA dynamo was derived for the first time, a number of its features were revealed. For example in [4, 5] the authors have extended the HSA dynamo including the effects of a nonlinear motor, an external battery and magnetic field, the coupling of two or more identical dynamos together. In subsequent works [19, 3, 15] and [16] the authors analysed the influence on the HSA model of nonlinearity of the series motor, the presence of a series battery and/or an ambient magnetic field. Bifurcation transition diagrams have been presented in [17], where the author also identifies unstable periodic orbits pertaining to some cases. The analysis towards identifying the underlying chaotic attractor was done in [18], where the author used some of the unstable periodic orbits to identify a possible template for the chaotic attractor, using ideas from topology.

The question whether a differential model admits a first integral (for the precise definition see Section 2) is of fundamental importance. One reason is that the first integrals give conservation laws for the model and that enables one to lower its dimension. Moreover, knowing a sufficient number of first integrals allows to solve the system explicitly. Finally the existence or non-existence [9, 14, 13] of first integrals for a given model measures, in a sense, the complexity of the set of its

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orbits. There are two equally difficult problems. One is to prove that a given system is chaotic; or to prove that it is not. In this paper we study the integrability problem for the HSA dynamo. We contribute to the understanding of the complexity, or more precisely to the topological structure of the orbits of HSA dynamo by studying the integrability problem for this model depending on its four parameters α, β, γ and κ . For proving our main results we shall use the information about invariant algebraic surfaces of this system, which is the basis of the so called Darboux theory of integrability, for more details see Section 2. We also note that for $\kappa = \lambda = 0$ and $\beta = 1$, the HSA dynamo is equivalent to the Nosé–Hoover equation (cf. [20] and [2]), for which the first integrals have been studied in [12].

In the following theorem we give explicit formulas for the first integrals of the HSA dynamo in the case that $\alpha \neq 0$.

Theorem 1. *Assume that $\alpha \neq 0$ and define the following functions:*

$$\begin{aligned}\mathcal{F}_1 &= \alpha \ln x - \alpha x^2/2 - y^2/2 + y; \\ \mathcal{F}_2 &= zv(x) - \int v(x)w(x) dx; \\ \mathcal{F}_3 &= \kappa \left[\log \frac{2(\kappa + y - 1 + \mathcal{T})}{-x} \right] - \mathcal{T}; \\ \mathcal{F}_4 &= \frac{y + \kappa - 1 - x\sqrt{\kappa}\sqrt{\kappa-1}}{y + \kappa - 1 + x\sqrt{\kappa}\sqrt{\kappa-1}} \exp \{2z\sqrt{\kappa}\sqrt{\kappa-1}\};\end{aligned}$$

where

$$w(x) = \left[1 - 2(\mathcal{F}_3 + \alpha \frac{x^2}{2} - \alpha \ln x) \right]^{-1/2}, \quad v(x) = \exp \left[\lambda \int xw(x) dx \right],$$

$$\mathcal{T} = \sqrt{-\kappa^2(x^2-1) + (y-1)^2 + \kappa(x^2+2y-2)}.$$

Then HSA dynamo admits the following first integral:

- (a) if $\beta = \kappa = 0$ and $\lambda \in \mathbb{R}$, then \mathcal{F}_1 and \mathcal{F}_2 are the first integrals;
- (b) if $\beta = 0$, $\alpha = -\kappa(\kappa-1)$ and $\kappa \neq 0$, then \mathcal{F}_3 is a first integral. Additionally, if $\lambda = 0$ then \mathcal{F}_4 is also a first integral.

It is a straightforward computation to check that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and \mathcal{F}_4 are the first integrals of system (1) for the corresponding values of the parameters, thus the proof of Theorem 1 will be omitted. One should not assume that it is trivial or easy to find those functions even though, once we know them, it is not difficult a check that they are first integrals. To date there are no general methods that would allow to decide whether a given system of differential equations is integrable or would give a way to calculate its first integrals. In the particular case of polynomial differential systems one of the best tools to approach this problem is the so-called Darboux theory of integrability. See Section 2 where we briefly explain how to use invariant algebraic surfaces (in general hypersurfaces) and exponential factors to construct a first integral.

Now we present other main results of the paper. The only values of the parameters of the HSA dynamo, for which it admits one or two Darboux first integrals (which include polynomial one) are given in Theorem 1. The following two theorems address the integrability problem for the HSA dynamo for the values of α, β, κ and λ not considered in Theorem 1. Thus they can be viewed as a non-integrability results. We consider two cases $\beta = 0$ (Theorem 1) and $\beta \neq 0$ (Theorem 2).

Theorem 2. *The following statements hold for the HSA dynamo equation with $\alpha \neq 0$, $\beta = 0$, $\kappa \neq 0$, $\alpha \neq -\kappa(\kappa-1)$ and $\lambda \in \mathbb{R}$:*

- (a) *It does not admit any polynomial first integral.*

- (b) *Its unique Darboux polynomial with nonzero cofactor is x .*
- (c) *Its only exponential factor is $\exp(z)$ with the cofactor $x - \lambda z$.*
- (d) *It is not Darboux integrable.*

Finally, we consider the case in which $\alpha, \beta \neq 0$ and $\kappa, \lambda \in \mathbb{R}$.

Theorem 3. *The following statements hold for the HSA dynamo equation with $\alpha, \beta \neq 0$, and $\kappa, \lambda \in \mathbb{R}$:*

- (a) *It does not admit any polynomial first integral.*
- (b) *It does not admit any Darboux polynomial.*
- (c) *Its only exponential factors are:*
 - (c.1) $\exp(z)$ and $\exp(-x^2/2 + y/\alpha - y^2/(2\alpha) - \beta z^2/2)$ with the cofactors $x - \lambda z$ and $y - 1$, respectively if $\kappa = \lambda = 0$,
 - (c.2) $\exp(z)$ with the cofactors $x - \lambda z$ in any other case.
- (d) *It is not Darboux integrable.*

Clearly, if system (1) does not admit a Darboux first integral, then it does not admit a rational first integral. Thus Theorem 2 and 3 imply the following simple corollary.

Corollary 4. *If $\alpha, \beta, \kappa \neq 0$, then the HSA dynamo does not admit any polynomial, rational or Darboux first integral.*

In Section 2 we present some preliminary results on Darboux theory of integrability that will be used all through the paper. In Section 3 we state some results on the HSA dynamo equation for $\alpha \neq 0$. Finally, Theorems 2 and 3 are proved in Sections 4 and 5, respectively.

2. PRELIMINARY RESULTS

The associated vector field to (1) is

$$(2) \quad \mathfrak{X} = [x(y-1) - \beta z] \frac{\partial}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial}{\partial y} + [x - \lambda z] \frac{\partial}{\partial z}.$$

Let $U \subset \mathbb{R}^3$ be an open subset. We say that the nonconstant function $H: U \rightarrow \mathbb{R}$ is a *first integral* of the polynomial vector field (2) associated to system (1), if $H(x(t), y(t), z(t)) = \text{constant}$ for all values of t for which the solution $(x(t), y(t), z(t))$ of \mathfrak{X} is defined on U . Clearly H is a first integral of \mathfrak{X} on U if and only if $\mathfrak{X}H = 0$ on U . When H is a polynomial we say that it is a *polynomial first integral*. We will say that system (1) is *integrable* if it admits a first integral. We will also say that this system is *completely integrable* if it admits two functionally independent first integrals. Since each level curve of a first integral is invariant under the flow induced by the system, it is clear that if this system is completely integrable, then the intersection of its two first integrals determines an invariant curve, which in turn gives information on the orbits of the system.

In what follows we recall the basic notion from the Darboux theory of integrability [7]. Let $h = h(x, y, z) \in \mathbb{C}[x, y, z]$ be a nonconstant polynomial. We say that $h = 0$ is an *invariant algebraic surface* of the vector field \mathfrak{X} if it satisfies $\mathfrak{X}h = Kh$ for some polynomial $K = K(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of $h = 0$ [8]. Note that K has degree at most 1. The polynomial h is called a *Darboux polynomial*, and we also say that K is the *cofactor* of the Darboux polynomial h . We note that a Darboux polynomial with zero cofactor is a polynomial first integral. Let $g, h \in \mathbb{C}[x, y, z]$ be coprime. We say that a nonconstant function $e^{g/h}$ is an *exponential factor* of the vector field \mathfrak{X} given in (2) if it satisfies $\mathfrak{X}e^{g/h} = Le^{g/h}$ for

some polynomial $L = L(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of $e^{h/g}$ and having degree at most 1. Note that this relation is equivalent to

$$(3) \quad [x(y-1) - \beta z] \frac{\partial(g/h)}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial(g/h)}{\partial y} + [x - \lambda z] \frac{\partial(g/h)}{\partial z} = L.$$

For a geometric and algebraic meaning of the exponential factors see [1]. A first integral G of system (1) is called of *Darboux type* or *Darboux first integral* if it is of the form

$$(4) \quad G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left[\exp \left(\frac{g_1}{h_1} \right) \right]^{\mu_1} \cdots \left[\exp \left(\frac{g_q}{h_q} \right) \right]^{\mu_q},$$

where f_j is a Darboux polynomial, $\left[\exp \left(\frac{g_k}{h_k} \right) \right]^{\mu_k}$ is an exponential factor, $g_k, h_k \in \mathbb{C}[x, y, z]$ and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \dots, p$, $k = 1, \dots, q$.

The Darboux theory of integrability gives a sufficient condition for the integrability using the information about the invariant algebraic hypersurfaces (in our case surfaces). In practise one does not need to exploit the information about all the invariant algebraic surfaces to construct a first integral [8]. It is enough to find any number, say p , of Darboux polynomials f_i with the cofactors K_i , and q exponential factors $\left[\exp \left(\frac{g_k}{h_k} \right) \right]^{\mu_k}$ with the corresponding cofactors L_k such that the linear combination

$$\sum_{j=1}^p \lambda_j K_j + \sum_{k=1}^q \mu_k L_k = 0,$$

where $\lambda_j, \mu_k \in \mathbb{C}$. Then the function G as in (4) is a first integral of \mathfrak{X} . For more information on the Darboux theory of integrability see for instance [7, 10, 11] and the references therein. Note that a polynomial or a rational first integral is a particular case of a Darboux first integral.

For a proof of the next proposition see [1].

Proposition 5. *The following statements hold.*

- (a) *If $E = e^{g/h}$ is an exponential factor for the polynomial system (1) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic curve.*
- (b) *Eventually e^g can be an exponential factor, coming from the multiplicity of the infinity.*

3. RESULTS OF THE HSA DYNAMO EQUATION WHEN $\alpha \neq 0$

Proposition 6. *System (1) when either $\alpha\beta \neq 0$ or $\alpha\kappa \neq 0$ does not admit a polynomial first integral.*

Proof. Let h be a polynomial first integral of system (1). Then it satisfies

$$(5) \quad [x(y-1) - \beta z] \frac{\partial h}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial h}{\partial y} + [x - \lambda z] \frac{\partial h}{\partial z} = 0.$$

Without loss of generality we can write

$$(6) \quad h = \sum_{j=1}^n h_j(x, y, z),$$

where each $h_j = h_j(x, y, z)$ is a homogeneous polynomial of degree j and we assume that $h_n \neq 0$ and $n \geq 1$.

Computing the terms of degree $n+1$ in (5) we get

$$(7) \quad xy \frac{\partial h_n}{\partial x} - \alpha x^2 \frac{\partial h_n}{\partial y} = 0.$$

Thus, solving this differential equation we get $h_n = h_n[z, (\alpha x^2 + y^2)/2]$. Since $h_n \neq 0$ is a homogeneous polynomial of degree $n \geq 1$, we conclude that

$$h_n = c_n z^{n-2m} (\alpha x^2 + y^2)^m,$$

for some nonnegative integer m and where c_n is a constant different from zero. We introduce the notation

$$(8) \quad \Gamma = \alpha x^2 + y^2.$$

Then $h_n = c_n z^{n-2m} \Gamma^m$. Now computing the terms of degree n in (5) we get

$$(9) \quad -(x + \beta z) \frac{\partial h_n}{\partial x} - \kappa y \frac{\partial h_n}{\partial y} + (x - \lambda z) \frac{\partial h_n}{\partial z} + xy \frac{\partial h_{n-1}}{\partial x} - \alpha x^2 \frac{\partial h_{n-1}}{\partial y} = 0.$$

Solving it with respect to h_{n-1} we obtain:

$$\begin{aligned} h_{n-1} = & \pm \frac{1}{\alpha^{1/2} \Gamma} z^{-1-2m} \left\{ c_n \Gamma^m z^n \left[-(2m-n)\Gamma - 2\alpha\beta m z^2 \right] \arctan\left(\frac{\alpha^{1/2} x}{y}\right) \right. \\ & + \alpha^{1/2} z \left(-2(\kappa-1)my + \Gamma^{1/2}(2\kappa m + \lambda(n-2m)) \right) \log\left(\frac{4(\Gamma y + \Gamma^{1/2})}{2\kappa m + \lambda(n-2m)x\Gamma^{3/2}z}\right) \\ & \left. + \alpha^{1/2} \Gamma z^{2m+1} c_{n-1}(z, \Gamma) \right\}, \end{aligned}$$

where c_{n-1} is a homogeneous polynomial in the variables z, Γ . Since h_{n-1} is a homogeneous polynomial of degree $n-1$ we must have

$$2m - n = 0, \quad \alpha\beta m = 0, \quad \alpha[2\kappa m + \lambda(n-2m)] = 0, \quad \alpha(\kappa-1)m = 0.$$

We have $n = 2m$. Furthermore, since $\alpha\beta \neq 0$ or $\alpha\kappa \neq 0$ we also have $m = 0$. This implies $n = 0$, a contradiction. \square

Proposition 7. *Let f be a Darboux polynomial of degree n with nonzero cofactor $K = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z$, $\beta_i \in \mathbb{C}$ for $i = 0, \dots, 3$ of system (1) with $\alpha \neq 0$. Then $\beta_1 = \beta_3 = 0$. Furthermore, if f_n is the homogeneous polynomial of degree $n \geq 1$ then (see (8) for Γ)*

$$(10) \quad f_n = c_n z^p x^{n-2m-p} \Gamma^m, \quad \beta_2 = n - p - 2m,$$

where m, p are some nonnegative integers and $c_n \in \mathbb{C} \setminus \{0\}$.

Proof. Let f be a Darboux polynomial of degree n with nonzero cofactor $K = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z$, $\beta_i \in \mathbb{C}$ for $i = 0, \dots, 3$ of system (1) with $\alpha \neq 0$. Then f satisfies

$$(11) \quad [x(y-1) - \beta z] \frac{\partial f}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial f}{\partial y} + [x - \lambda z] \frac{\partial f}{\partial z} = (\beta_0 + \beta_1 x + \beta_2 y + \beta_3 z)h.$$

It is easy to see by direct computations that h has degree greater than or equal to two since system (1) has no Darboux polynomials of degree one with nonzero cofactor. Thus we decompose f as a sum of homogeneous polynomials similarly as in (6), where $n \geq 1$ and $f_n \neq 0$.

Computing the terms of degree $n+1$ in (11) we get

$$xy \frac{\partial f_n}{\partial x} - \alpha x^2 \frac{\partial f_n}{\partial y} = (\beta_1 x + \beta_2 y + \beta_3 z) f_n.$$

Solving this linear differential equation we obtain

$$f_n = \exp \left[\pm \frac{\beta_1}{\alpha^{1/2}} \arctan\left(\frac{\alpha^{1/2} x}{y}\right) \right] \left(\frac{-2(\Gamma + y\Gamma^{1/2})}{\beta_3 x z \Gamma^{1/2}} \right)^{\pm \frac{\beta_3 z}{\Gamma}} x^{\beta_2} c_n(z, \Gamma),$$

where c_n is a function in the variables z and Γ . Since f_n is a homogeneous polynomial of degree n we must have $\beta_1 = \beta_3 = 0$. Furthermore

$$f_n = c_n z^p x^{n-2m-p} \Gamma^m, \quad \beta_2 = n - p - 2m,$$

where m, p are some nonnegative integers and $c_n \in \mathbb{C} \setminus \{0\}$. \square

Proposition 8. *Let $g \in \mathbb{C}[x, y, z]$ satisfy*

$$(12) \quad [x(y-1) - \beta z] \frac{\partial g}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial g}{\partial y} + (x - \lambda z) \frac{\partial g}{\partial z} = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z,$$

where $\beta_i \in \mathbb{C}$, for $i = 0, 1, 2, 3$ and not all zero and $\alpha \neq 0$. Then

- (1) *If $\kappa \neq 0$ or $\kappa = 0$ and $\beta, \lambda \neq 0$ then the unique solution is $h = \beta_1 z$ with $L = \beta_1(x - \lambda z)$.*
- (2) *If $\kappa = 0$ and $\lambda = 0$ then we obtain two solutions: $h = \beta_1 z$ with $L = \beta_1 x$ and $h = -\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha} - \frac{\beta}{2} z^2$ with $L = \beta_0(1 - y)$.*
- (3) *If $\kappa = 0$, $\lambda \neq 0$ and $\beta = 0$ then we obtain two solutions: $h = \beta_1 z$ with $L = \beta_1(x - \lambda z)$ and $h = -\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha}$ with $L = \beta_0(1 - y)$.*

Proof. We first prove that g is a polynomial of degree two. We proceed by contradiction. Assume that g is polynomial of degree $n \geq 3$. We write it as a sum of its homogeneous parts as in equation (6) with h_j replaced by g_j . Without loss of generality we can assume that $g_n \neq 0$. Then since the right-hand side of equation (12) has degree at most one, computing the terms of degree $n+1$ in equation (12) we get

$$xy \frac{\partial g_n}{\partial x} - \alpha x^2 \frac{\partial g_n}{\partial y} = 0,$$

which is equation (7) replacing h_n by g_n . Then the arguments used in the proof of Proposition 6 imply that n must be even and that g_n must be of the form $g_n = \alpha_n(\alpha x^2 + y^2)^{n/2}$ with $\alpha_n \in \mathbb{C} \setminus \{0\}$.

Now computing the terms in (12) of degree $n \geq 3$ and taking into account that the right-hand side of (12) has degree one, we get equation

$$-(x + \beta z) \frac{\partial g_n}{\partial x} - \kappa y \frac{\partial g_n}{\partial y} + (x - \lambda z) \frac{\partial g_n}{\partial z} + xy \frac{\partial g_{n-1}}{\partial x} - \alpha x^2 \frac{\partial g_{n-1}}{\partial y} = 0.$$

which is equation (9) with h_n replaced by g_n and h_{n-1} replaced by g_{n-1} . The arguments used in the proof of Proposition 6 imply that $g_n = 0$. Then we have that $g_n = 0$ for $n \geq 3$, and thus, g is a polynomial of degree at most two satisfying (12). Without loss of generality we can assume that it has no constant term. Then we write it as

$$g_n = g_{100}x + g_{010}y + g_{001}z + g_{200}x^2 + g_{110}xy + g_{101}xz + g_{020}y^2 + g_{011}yz + g_{002}z^2,$$

where $g_{ijk} \in \mathbb{C}$ for $0 \leq i, j, k \leq 2$. Then

$$\begin{aligned} & [x(y-1) - \beta z] (g_{100} + 2g_{200}x + g_{110}y + g_{101}z) \\ & + [\alpha(1-x^2) - \kappa y] (g_{010} + g_{110}x + 2g_{020}y + g_{011}z) \\ & + (x - \lambda z) (g_{001} + g_{101}x + g_{011}y + 2g_{002}z) \\ & = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z. \end{aligned}$$

Solving this equation we get:

Case 1: $\kappa \neq 0$ or $\kappa = 0$ and $\beta, \lambda \neq 0$. In this case the unique solution is $h = \beta_1 z$ with cofactor $L = \beta_1(x - \lambda z)$.

Cases 2: $\kappa = 0$ and $\lambda = 0$ Then we obtain two solutions: $h = \beta_1 z$ with $L = \beta_1 x$ and $h = -\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha} - \frac{\beta}{2} z^2$ with $L = \beta_0(1 - y)$.

Case 3: $\kappa = 0$, $\lambda \neq 0$ and $\beta = 0$. We obtain two solutions: $h = \beta_1 z$ with $L = \beta_1(x - \lambda z)$ and $h = -\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha}$ with $L = \beta_0(1 - y)$.

This completes the proof of the proposition. \square

4. PROOF OF THEOREM 2

The proof of Statement a) follows directly from Proposition 6.

To prove Statement b) we first note that if h is a Darboux polynomial with nonzero cofactor then it is easy to see by direct computations that the unique Darboux polynomial with degree one is x . Now we assume that h is an irreducible Darboux polynomial of degree n of system (1) with nonzero cofactor $K = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z$ with $\beta_i \in \mathbb{C}$ for $i = 0, \dots, 3$ not all zero and that has degree greater than or equal to two. Thus we decompose h as a sum of homogeneous polynomials similarly as in (6), where $n \geq 2$ and $h_n \neq 0$. By Proposition 7 we get $\beta_1 = \beta_3 = 0$ and

$$(13) \quad h_n = c_n z^p x^{n-p-2m} \Gamma^m, \quad c_n \in \mathbb{C} \setminus \{0\},$$

and $\beta_2 = n - p - 2m$.

Since h is irreducible and $x = 0$ is invariant, if we restrict h to $x = 0$ and denote it by \bar{h} we must have that $\bar{h} \neq 0$. Furthermore \bar{h} satisfies

$$(\alpha - \kappa y) \frac{\partial \bar{h}}{\partial y} - \lambda z \frac{\partial \bar{h}}{\partial z} = (\beta_0 + \beta_2 y) \bar{h}.$$

Hence, solving this equation we obtain

$$\bar{h} = e^{-\beta_2 y / \kappa} (\kappa y - \alpha)^{-\frac{\alpha \beta_2 + \beta_0 \kappa}{\kappa^2}} C[z(\kappa y - \alpha)^{-\lambda / \kappa}],$$

where C is a function in the variable $z(\kappa y - \alpha)^{-\lambda / \kappa}$. Since \bar{h} must be a homogeneous polynomial we must have $\beta_2 = 0$, i.e., $n = p + 2m$. Hence

$$(14) \quad \bar{h} = (\kappa y - \alpha)^{-\beta_0 / \kappa} P[z(\kappa y - \alpha)^{-\lambda / \kappa}],$$

where P is a polynomial in the variable $z(\kappa y - \alpha)^{-\lambda / \kappa}$, and the

$$h_n = c_n z^p (\alpha x^2 + y^2)^m, \quad c_n \in \mathbb{C} \setminus \{0\}.$$

Computing the terms of degree n in (11) we get

$$(15) \quad -x \frac{\partial h_n}{\partial x} - \kappa y \frac{\partial h_n}{\partial y} + (x - \lambda z) \frac{\partial h_n}{\partial z} + xy \frac{\partial h_{n-1}}{\partial x} - \alpha x^2 \frac{\partial h_{n-1}}{\partial y} = \beta_0 h_n.$$

Substituting (13) into equation (15) and solving it with respect to h_{n-1} we obtain

$$\begin{aligned} h_{n-1} = & \pm \frac{1}{\sqrt{\alpha} \Gamma z} \left\{ c_n \Gamma^m z^p \left(p \Gamma \arctan \left(\frac{\sqrt{\alpha} x}{y} \right) - z \sqrt{\alpha} [2(\kappa - 1) m y \right. \right. \\ & \left. \left. - (\beta_0 + 2\kappa m + \lambda p) \sqrt{\Gamma} \log \left(\frac{-4(\Gamma^{1/2} + y)}{(\beta_0 + 2\kappa m + \lambda p) x \Gamma z} \right) \right] \right) \\ & \left. + \alpha^{1/2} \Gamma z c_{n-1}(z, \Gamma) \right\}, \end{aligned}$$

where c_{n-1} is a function in the variables z, Γ . Since h_{n-1} is a homogeneous polynomial of degree $n - 1$ and $c_n \alpha \neq 0$ we must have

$$p = 0, \quad \beta_0 + 2\kappa m + \lambda p = 0.$$

From $p = 0$ we deduce that $m = n/2$ (n must be even) and $\beta_0 = -n\kappa$. Then

$$(16) \quad h_{n-1} = \pm c_n n (\kappa - 1) (\alpha x^2 + y^2)^{n/2-1} y + c_{n-1} z^{n-1-2l} (\alpha x^2 + y^2)^l,$$

for some $c_{n-1} \in \mathbb{C}$ and some nonnegative integer l . Also h_n assumes now the simplified form

$$h_n = c_n \Gamma^{n/2} = c_n (\alpha x^2 + y^2)^{n/2}, \quad c_n \in \mathbb{C} \setminus \{0\}.$$

Computing the terms of degree $n-1$ in (11) we get

$$\begin{aligned} h_{n-2} = & \left\{ 2c_{n-1}(n-1-2l)z^n \Gamma^{l+2} \arctan\left(\frac{x\alpha^{3/2}}{y}\right) \right. \\ & + \alpha^{1/2} 4c_{n-1} l y z^{n+1} \Gamma^{1+l} (1-\kappa) + c_n (2-n) n x^2 z^{2+2l} \alpha \Gamma^{n/2} (\kappa-1)^2 \\ & - \alpha^{1/2} 2c_n n z^{2+2l} \Gamma^{1+n/2} (\alpha + (\kappa-1)\kappa) \log(x) \\ & \left. - 2\alpha^{1/2} c_{n-1} z^{n+1} \Gamma^{3/2+l} [(\kappa-\lambda)(n-2l) + \lambda] \log(\Theta) \right\} + c_{n-2}[z, \Gamma], \end{aligned}$$

where c_{n-2} is a function of z, Γ and $\Theta = 4(y + \Gamma^{1/2})/[(\kappa-\lambda)(n-2l) + \lambda]$. Since h_{n-2} is a homogeneous polynomial of degree $n-2$ we must necessary have

$$c_{n-1}(n-1-2l) = 0.$$

If $n-1-2l = 0$, then again, since h_{n-2} is polynomial we must have $(\kappa-\lambda)(n-2l) + \lambda = 0$, which implies that $\kappa = 0$ and this is in contradiction with our assumption. On the other hand if $c_{n-1} = 0$, then calculating then

$$h_{n-2} = -\frac{1}{2} c_n n \Gamma^{n/2-2} \left[(n-2)x^2 \alpha (\kappa-1)^2 + 2\Gamma[\alpha + \kappa(\kappa-1)] \log(x) \right] + c_{n-2}[z, \Gamma].$$

Thus since h_{n-2} is a polynomial, we get that $\alpha + \kappa(\kappa-1) = 0$, which is in contradiction with our assumption. This concludes the proof of statement b) in the theorem.

To prove Statement c) we note that in view of Proposition 5 if E is an exponential factor of system (1) with $\beta = 0$, $\alpha \neq 0$ then it is of the form

$$E = e^{g/x^n},$$

for some nonnegative integer n ; and g and x^n are coprime. Then g satisfies the equation

$$x(y-1) \frac{\partial g}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial g}{\partial y} + (x-\lambda z) \frac{\partial g}{\partial z} - n(y-1)g = (\beta_0 + \beta_1 x + \beta_2 y + \beta_3 z)x^n,$$

where we have simplified by the common factor $x^n e^{g/x^n}$ and where $\beta_i \in \mathbb{C}$ for $i = 0, \dots, 3$. We consider two different cases.

Case 1: $n \geq 1$. In this case if we denote by \bar{g} the restriction of g to $x = 0$ we have that $\bar{g} \neq 0$ (otherwise g would be divisible by x , which is not possible) and \bar{g} satisfies

$$(\alpha - \kappa y) \frac{\partial \bar{g}}{\partial y} - \lambda z \frac{\partial \bar{g}}{\partial z} = n(y-1)\bar{g}.$$

Hence, \bar{g} is a Darboux polynomial of system (1) with $\beta = 0$, $\kappa \neq 0$ and restricted to $x = 0$. Solving this partial differential equation we obtain

$$\bar{g} = e^{-\frac{ny}{\kappa}} (\kappa y - \alpha)^{\frac{n(\kappa-\alpha)}{\kappa^2}} C[z(\alpha - \kappa y)^{-\lambda/\kappa}]$$

Since \bar{g} must be a polynomial and $n\kappa \neq 0$ we have that $\bar{g} = 0$, a contradiction. Hence this case is not possible.

Case 2: $n = 0$. In this case $E = e^g$ where g satisfies (12). In view of Proposition 8 and since $\beta = 0$ with $\kappa \neq 0$ we obtain that the unique exponential factors are e^z with cofactor $x - \lambda z$. This concludes the proof of the statement c).

The proof of statement d) will be done by contradiction. Assume that G is a first integral of Darboux type. In view of the definition of first integral of Darboux type in (4) and taking into account statements a), b) and c), G must be of the form

$$G = x^\lambda e^{\mu z}, \quad \text{with } \lambda, \mu \in \mathbb{C}.$$

Since G is a first integral it must satisfy $\mathfrak{X}G = 0$, that is,

$$\begin{aligned} \mathfrak{X}G &= x(y-1)\frac{\partial G}{\partial x} + [\alpha(1-x^2) - \kappa y]\frac{\partial G}{\partial y} + (x - \lambda z)\frac{\partial G}{\partial z} \\ &= [\lambda(y-1) + \mu(x - \lambda z)]G = 0. \end{aligned}$$

Hence, $\lambda(y-1) + \mu(x - \lambda z) = 0$, which implies $\lambda = \mu = 0$. Then $G = \text{constant}$, in contradiction with the fact that G was a first integral. This concludes the proof of the theorem.

5. PROOF OF THEOREM 3

The proof of Statement a) follows directly from Proposition 6.

To prove Statement b) we first note that if h is a Darboux polynomial with nonzero cofactor then it is easy to see by direct computations that h has degree greater than or equal to two since system (1) has no Darboux polynomials of degree one with nonzero cofactor $K = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z$ with $\beta_i \in \mathbb{C}$ for $i = 0, \dots, 3$ not all zero. Thus we decompose h as a sum of homogeneous polynomials similarly as in (6), where $n \geq 2$ and $h_n \neq 0$. By Proposition 7 we get $\beta_1 = \beta_3 = 0$ and $h_n = c_n z^p x^{n-p-2m} \Gamma^m$, $c_n \in \mathbb{C} \setminus \{0\}$ and $\beta_2 = n - p - 2m$.

Computing the terms of degree n in (11) we get

$$\begin{aligned} (17) \quad & - (x + \beta z) \frac{\partial h_n}{\partial x} - \kappa y \frac{\partial h_n}{\partial y} + (x - \lambda z) \frac{\partial h_n}{\partial z} + xy \frac{\partial h_{n-1}}{\partial x} - \alpha x^2 \frac{\partial h_{n-1}}{\partial y} \\ & = \beta_0 h_n + (n - p - 2m) y h_{n-1}. \end{aligned}$$

Substituting (10) into equation (17) and solving it with respect to h_{n-1} we obtain

$$\begin{aligned} (18) \quad h_{n-1} &= \pm \frac{x^{-1-2m+n-p}}{\sqrt{\alpha} \Gamma z} \left\{ c_n \Gamma^m z^p \left(x(\alpha p x^2 + p y^2 - 2\alpha \beta m z^2) \arctan\left(\frac{\sqrt{\alpha} x}{y}\right) \right. \right. \\ &+ \sqrt{\alpha} z (y(-2(\kappa-1)m x + \beta(-2m+n-p)z) \\ &+ [\beta_0 + 2(\kappa-1)m + n + (\lambda-1)p] x \sqrt{\Gamma} \log\left(\frac{-4(\Gamma + y \Gamma^{1/2})}{(\beta_0 + 2(\kappa-1)m + n + (\lambda-1)p) x \Gamma^{3/2} z}\right) \\ &\left. \left. + \sqrt{\alpha} x \Gamma z c_{n-1}(z, \Gamma) \right\} \right\} \end{aligned}$$

where $c_{n-1}(z, \Gamma)$ is a function of the variables z, Γ . Since h_{n-1} is a homogeneous polynomial of degree $n-1$ we must have in particular that

$$p = 0, \quad \alpha \beta m = 0, \quad \beta_0 + 2(\kappa-1)m + n + (\lambda-1)p = 0.$$

Since $\alpha \beta \neq 0$ this implies $p = m = 0$ and $\beta_0 = -n$. Hence h_{n-1} becomes

$$h_{n-1} = \frac{c_n \beta n y z}{\Gamma} x^{n-1} + x^n c_{n-1}(z, \Gamma).$$

Again since h_{n-1} must be a homogeneous polynomial of degree $n-1$ and $\alpha \beta c_n \neq 0$ we must have $n = 0$. But then $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$ a contradiction with the fact that h was a Darboux polynomial with nonzero cofactor. This concludes the proof of statement b) in the theorem.

To prove statement c) we note that from Proposition 5 we can write $E = e^g$ and g satisfies (12). Then statement c) follows now directly from Proposition 8.

The proof of statement d) will be done by contradiction. Assume that G is a first integral of Darboux type. We consider three different cases:

Case 1: $\kappa \neq 0$ or $\kappa = 0$ and $\lambda\beta \neq 0$. Then in view of the definition of first integral of Darboux type in (4) and taking into account statements a), b) and c), G must be of the form

$$G = e^{\mu z}, \quad \text{with } \mu \in \mathbb{C}.$$

Since G is a first integral it must satisfy $\mathfrak{X}G = 0$, that is,

$$\begin{aligned} \mathfrak{X}G &= (x(y-1) - \beta z) \frac{\partial G}{\partial x} + (\alpha(1-x^2) - \kappa y) \frac{\partial G}{\partial y} + (x - \lambda z) \frac{\partial G}{\partial z} \\ &= \mu(x - \lambda z)G = 0. \end{aligned}$$

Hence, $\mu(x - \lambda z) = 0$, which implies $\mu = 0$. Then $G = \text{constant}$, in contradiction with the fact that G was a first integral.

Case 2: $\kappa = 0$ and $\lambda = 0$. In this case in view of the definition of first integral of Darboux type in (4) and taking into account statements a), b) and c), G must be of the form

$$G = e^{\mu_1 z} e^{\mu_2 (-\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha} - \frac{\beta}{2} z^2)}, \quad \text{with } \mu_1, \mu_2 \in \mathbb{C}.$$

Since G is a first integral it must satisfy $\mathfrak{X}G = 0$, that is,

$$\begin{aligned} \mathfrak{X}G &= [x(y-1) - \beta z] \frac{\partial G}{\partial x} + [\alpha(1-x^2) - \kappa y] \frac{\partial G}{\partial y} + [x - \lambda z] \frac{\partial G}{\partial z} \\ &= [\mu_1(x - \lambda z) + \mu_2(1 - y)]G = 0. \end{aligned}$$

Hence, $\mu_1(x - \lambda z) + \mu_2(1 - y) = 0$, which implies $\mu_1 = \mu_2 = 0$. Then $G = \text{constant}$, in contradiction with the fact that G was a first integral.

Case 3: $\kappa = 0$, $\lambda \neq 0$ and $\beta = 0$. Then in view of the definition of first integral of Darboux type in (4) and taking into account statements a), b) and c), G must be of the form

$$G = e^{\mu_1 z} e^{\mu_2 (-\frac{x^2}{2} + \frac{y}{\alpha} - \frac{y^2}{2\alpha})}, \quad \text{with } \mu_1, \mu_2 \in \mathbb{C}.$$

Since G is a first integral it must satisfy $\mathfrak{X}G = 0$, that is,

$$\mathfrak{X}G = (\mu_1(x - \lambda z) + \mu_2(1 - y))G = 0.$$

Proceeding as in the case above we reach a contradiction. This concludes the proof of the theorem.

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